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ZERO DISTRIBUTION FOR PAIRS OF HOLOMORPHIC FUNCTIONS WITH APPLICATIONS TO EIGENVALUE DISTRIBUTION

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ABSTRACT. Let f and g be holomorphic in an angle Λ. Theorem 1 shows that the zero-distributions of f and g are comparable if, near $\partial\Lambda$, f and g grow similarly. This result is applied to analyse the asymptotic behavior of eigenvalues of certain perturbed normal operators.

1. Introduction. For $\frac{1}{2} < \alpha < \infty$ let

$$\Lambda = \Lambda_\alpha = \{z; |\arg z| \leqslant \pi/2\alpha\}, \tag{1.1}$$

and let f and g be holomorphic in (the closure of) Λ_α. If Λ were bounded, Roché's theorem would guarantee that f and g have the same number of zeros in Λ if, say, $|f(z) - g(z)| < g(z)$ for $z \in \partial\Lambda$. Our main result provides a way of obtaining similar conclusions for Λ as defined in (1.1); the method is to compare the behavior of f and g near $\partial\Lambda$ to that of an appropriate regularly varying auxiliary function.

The auxiliary functions considered will be nonnegative, nondecreasing functions defined for $t > 0$, with their behavior limited as $t \to \infty$. If $\phi(t)$ and $\psi(t)$ are nonnegative and nondecreasing for $t > 0$, we say

$$\phi \asymp \psi \tag{1.2}$$

if $|\psi(t)/\phi(t)|^{\pm 1} = O(1)$ $(t \to \infty)$, and

$$\phi \sim \psi \tag{1.3}$$

if $\phi(t) = \{1 + o(1)\}\psi(t)$ $(t \to \infty)$.

For a fixed $p > 0$, the class $A(p)$ consists of nonnegative, nondecreasing functions $\phi(t)$ $(t > 0)$ such that there exists $a > 1$ such that

$$\phi(at) \leqslant a^p\phi(t) \qquad (t > t_0(a)), \tag{1.4}$$

and we set

$$A = \bigcup A(p) \qquad (p > 0); \tag{1.5}$$

in (1.4) and below, the qualification $t > t_0(\alpha, \beta, \dots)$ means: when t is sufficiently large; this bound may depend on the parameters $\alpha, \beta, \dots$, and the choice of t_0 is not

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necessarily the same at each occurence. Condition (1.4) implies that ϕ has no Pólya-peaks of order $> p$ [7, p. 101] and an immediate consequence of (1.4) is that

$$\phi(y) \leqslant C(y/t)^p \phi(t) \qquad (t_0 \leqslant t \leqslant y,\ C = C(a)). \tag{1.6}$$

The class B consists of functions $\phi(t)$ $(t > 0)$ which are nonnegative, nondecreasing, such that for some $a > 1$,

$$\phi(at) \geqslant 2\phi(t) \qquad (t > t_0(a)). \tag{1.7}$$

Finally, if given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\phi((1+\delta)t) \leqslant (1+\varepsilon)\phi(t) \qquad (t > t_0), \tag{1.8}$$

then we say $\phi \in C$. Clearly, $C \subset A$.

In order to state our main result, we first recall that if f is analytic in the sector $\Lambda = \Lambda_\alpha$ of (1.1), and if $M(r, f, \Lambda) = \max |f(z)|$ $(|z| = r,\ z \in \Lambda)$, then the *order* ρ of f is

$$\rho = \limsup_{r \to \infty} \frac{\log\log M(r, f, \Lambda)}{\log r}. \tag{1.9}$$

The zeros of f are counted by the usual functionals

$$n_f(r) = \sum_{\substack{|z_k| < r \\ z_k \in \bar{\Lambda}}} 1, \qquad N_f(r) = \int_0^r n_f(t) t^{-1}\, dt,$$

where the z_k are the roots of f in the interior of Λ. We assume throughout that $n_f(r)$ and $n_g(r)$ are zero for $0 \leqslant r < h$ for some fixed $h > 0$.

THEOREM 1. *Let f and g be holomorphic of order $\leqslant \alpha$ in the sector Λ_α, such that all their zeros are contained in $\Lambda_{\alpha+\tau}$ for some $\tau > 0$. Suppose the zeros of g are regularly distributed in the sense that $n_g(r) \in B \cap A(p)$* (*cf.* (1.3), (1.4)) *for some $p < \alpha$, and that*

$$\big|\ln|f(z)/g(z)|\big| = o\big(n_g(r)\big) \tag{1.10}$$

uniformly as $z \to \infty$ in $\Lambda_\alpha - \Lambda_{\alpha+\tau}$. Then, using the notation (1.2),

$$N_g(r) \asymp N_f(r), \qquad n_g(r) \asymp n_f(r). \tag{1.11}$$

Note 1. In Theorem 2, in §4, we shall modify Theorem 1 to obtain conditions needed to deduce that $N_g \sim N_f$ and $n_g \sim n_f$.

Note 2. In general, equivalence of N_f and N_g is not equivalent to that of n_f and n_g; this is where the assumption $n_g \in B \cap A(p)$ is needed. We remark that if $n(t) \in B$, then

$$N(r) \equiv \int_0^r n(t) t^{-1}\, dt \leqslant Cn(r), \tag{1.12}$$

and if $n(t) \in A$ then

$$n(r) \leqslant CN(r). \tag{1.13}$$

We sketch the proofs. If $n \in B$, take $a > 1$ so that (1.7) holds. For $r > t_0$, choose j with $a^j t_0 \leqslant r \leqslant a^{j+1}t_0$; then

$$N(r) \leqslant \int_0^{t_0} n(t)t^{-1}\,dt + \sum^{j} \int_{a^k t_0}^{a^{k+1}t_0} n(t)t^{-1}\,dt$$
$$= O(1) + O(\log a)n\left(a^{j+1}t_0\right) \leqslant Cn(r).$$

The proof of (1.3) is easier, using (1.6):

$$N(r) \geqslant \int_{r/2}^{r} n(t)t^{-1}\,dt \geqslant Cn\left(\frac{r}{2}\right)\int_{r/2}^{r}\left(\frac{r}{t}\right)^p t^{-1}\,dt \geqslant Cn(r).$$

2. Preliminary remarks and lemmas. By replacing z by $z^{2(2\alpha+\tau)^{-1}}$, we may assume $\alpha = 1 - \delta$, with $\delta = \tau(2\alpha + \tau)^{-1}$, and that (1.10) holds in $\Lambda_{1-\delta} - \Lambda_{1+\delta}$. The hypothesis of Theorem 1 now is that $n_g(r) \in B \cap A(p')$, where $p' = 2p(2\alpha + \tau)^{-1} < 1$ and that both f and g have order $\leqslant 1 - \delta$ in $\Lambda_{1-\delta}$. This normalization will be in effect through §4.

LEMMA 1. *Let $\delta > 0$ be as just described. Then, given $\varepsilon > 0$, we have*

$$n_f(r) + n_g(r) \leqslant r^{1-\delta+\varepsilon} \qquad (r > r_0). \tag{2.1}$$

PROOF. The function $f_1(z) = f(z^{1/(1-\delta)})$ is holomorphic in Λ_1, with zeros in $\Lambda_{1+\delta'}$ (where $\delta' = 2\delta(1-\delta)^{-1}$) and f_1 has order $\leqslant 1$. Since order $(f_1) \leqslant 1$, we obtain from [9, Theorem 4, Chapter 4] that $\nu_{f_1}(r) \leqslant r^{1+\varepsilon/2}$ $(r > r_0)$, where $\nu_{f_1}(r)$ is the number of zeros of f_1 in $\{|z - r/2| < r/2\}$. But all zeros of f_1 are in $\Lambda_{1+\delta'}$, so $n_{f_1}(r) \leqslant C\nu_{f_1}(r) \leqslant r^{1+\varepsilon}$ $(r > r_0)$, and since $n_f(r) \leqslant (n_{f_1}(r))^{(1-\delta)}$, we have estimate (2.1) for $n_f(r)$. Similarly, $n_g(r) = O(r^{1-\delta+\varepsilon})$ $(r \to \infty)$, and (2.1) follows.

COROLLARY. *Let z_k be the zeros of f. Then the Blaschke product with zero set z_k converges in the plane*

$$B(z) = B_f(z) = \prod (z_k - z)/(z_k + z). \tag{2.2}$$

The Blaschke product made from the zeros of g, $B_g(z)$, also converges. Both products represent bounded analytic functions in Λ_1 and meromorphic functions in the plane.

PROOF. Lemma 1 implies that the Blaschke condition $\Sigma(\operatorname{Re} z_k)/(1 + |z_k|^2) < \infty$ holds.

We can estimate the Nevanlinna characteristic [7] of B_g; only later, in Lemma 9, we can obtain an equivalent estimate for B_f.

LEMMA 2. *The meromorphic function B_g satsifies*

$$T(r, B_g) \leqslant Cn_g(r) = O(r^{1-\delta+\varepsilon}). \tag{2.3}$$

PROOF. Let z_k be the zeros of g in $\Lambda_{1+\delta}$ and

$$P_1(z) = \prod(1 - z/z_k), \qquad P_2(z) = \prod(1 + z/z_k);$$

Lemma 1 implies that both products converge to a function of order < 1. We can say a little more: since $n(r, P_i) = n_g(r) \in A_p$ $(i = 1, 2)$, the usual estimate for a canonical product [7, p. 27, formula (1.21) with $q = 1$] and (1.6) show

$$\ln|P_i(z)| \leq \int_0^r \frac{n_g(t)}{t}\,dt + r\int_r^\infty \frac{n_g(t)}{t^2}\,dt$$
$$\leq N_g(r) + Cr^{1-p}n_g(r)\int_r^\infty t^{p-2}\,dt \leq Cn_g(r).$$

Since $B_g = P_1P_2^{-1}$, Nevanlinna's first fundamental theorem implies that

$$T(r, B_g) \leq T(r, P_1) + T(r, P_2) + O(1) \leq Cn_g(r),$$

which, with Lemma 1, is (2.3).

COROLLARY. *The characteristic of the Blaschke product B_f satisfies*

$$T(r, B_f) = O(r^{1-\delta+\varepsilon}). \tag{2.4}$$

REMARK. Theorem 1 implies the more refined result that (2.3) holds with B_f in place of B_g.

PROOF. Define P_1 and P_2 as in the proof of Lemma 2, now with z_k the zeros of f. Estimate (2.1) on $n_f(t)$ may be used to estimate $T(r, P_1)$ and $T(r, P_2)$ as was done above. We thus obtain $T(r, B_f) \leq T(r, P_1) + T(r, P_2) + O(1) = O(r^{1-\delta+\varepsilon})$, which is (2.4).

LEMMA 3. *Let $F(z)$ be holomorphic in Λ_1, with all zeros in $\Lambda_{1+\delta}$ for some $\delta > 0$. Then*

$$N_F(r) = \frac{1+\delta}{\pi^2\delta}\int_0^r\left[J_F\left(\frac{\pi}{2}, t\right) - J_F\left(\frac{\pi}{2(1+\delta)}, t\right) - J_F\left(-\frac{\pi}{2}, t\right) + J_F\left(-\frac{\pi}{2(1+\delta)}, t\right)\right]\frac{dt}{t}$$
$$+ \frac{2(1+\delta)^2}{\pi^3\delta^2}\int_{\pi/2(1+\delta)}^{\pi/2}\int_{-\pi/2}^{-\pi/2(1+\delta)}\int_\psi^\theta \ln|F(re^{i\phi})|\,d\phi\,d\psi\,d\theta,$$

where

$$J_F(\phi, t) = \int_0^t \ln|F(se^{i\phi})|s^{-1}\,ds.$$

PROOF. This follows from the generalized Jensen formula as presented in [9, p. 143]. Let $n_F(t, \psi, \theta)$ be the number of zeros of F inside $S(t, \psi, \theta) \equiv \{\psi < \arg z < \theta\} \cap \{|z| < t\}$, and $N_F(r, \psi, \theta) = \int_0^r n_F(t, \psi, \theta)t^{-1}\,dt$.

If $F \neq 0$ on $\partial S(t, \psi, \theta)$, then the generalized Jensen formula implies that

$$N_F(r, \psi, \theta) = \frac{1}{2\pi}\left[\frac{d}{d\phi}\int_0^r J_F(\phi, t)t^{-1}\,dt\right]_{\phi=\theta} - \frac{1}{2\pi}\left[\frac{d}{d\phi}\int_0^r J_F(\phi, t)t^{-1}\,dt\right]_{\phi=\psi} + \frac{1}{2\pi}\int_\psi^\theta \ln|F(re^{i\phi})|\,d\phi. \tag{2.5}$$

Lemma 3 follows from (2.5) on integrating with respect to ψ and θ, and recalling that all zeros of F are in $\Lambda_{1+\delta}$.

We may use Lemma 3 to give a representation for $N_f(r) - N_g(r)$. Let B_f and B_g be the Blaschke products made from the zeros of f and g, and

$$B^*(z) = B_f(z)/B_g(z), \tag{2.6}$$

so that B^* is meromorphic (Corollary to Lemma 1). Since $|B^*(iy)| = 1$, we obtain from Lemma 3:

LEMMA 4. *The counting functions of f and g are related by*

$$N_f(r) - N_g(r) = \frac{1+\delta}{\pi^2\delta}\int_0^r\left[I\left(\frac{\pi}{2(1+\delta)}, t\right) - I\left(-\frac{\pi}{2(1+\delta)}, t\right)\right]t^{-1}\,dt + \frac{2(1+\delta)^2}{\pi^3\delta^2}\int_{\pi/2(1+\delta)}^{\pi/2}\int_{-\pi/2}^{-\pi/2(1+\delta)}\int_{\psi}^{\theta}\ln|B^*(re^{i\phi})|\,d\phi\,d\psi\,d\theta, \tag{2.7}$$

where

$$I(\phi, t) = \int_0^t \ln|B^*(se^{i\phi})|s^{-1}\,ds.$$

The proof of Theorem 1 will follow on estimating the terms on the right side of (2.7). This will be done in §3. Our final results here seem to be new, although Lemma 6 has also been obtained by G. V. Radzievskii (unpublished).

LEMMA 5. *Let F be holomorphic in Λ_α, of order $< \alpha$, and suppose*

$$\ln|F(z)| \leqslant \phi(r) \qquad (z \in \partial\Lambda_\alpha) \tag{2.8}$$

where $\phi \in A(p)$ for some $p < \alpha$ (cf. (1.4)). Then

$$\ln|F(z)| \leqslant C\phi(|z|) \qquad (z \in \Lambda_\alpha) \tag{2.9}$$

for some constant C.

PROOF. This lemma is similar to a result of Beurling [3, p. 34]. However, we assume that ϕ has order strictly less than α, and satisfies (1.4), but in turn deduce the global conclusion (2.9).

Consider $\Phi(z) = F(z)e^{-\varepsilon z^p - b}$ in Λ_α, where we take $x^p > 0$ if $x > 0$. The constants b and ε are determined as follows. Let t_0 be associated to f by (1.4), let $r > t_0$; we take $b = f(r)$ and $\varepsilon = a^p f(r) r^{-p}\cos^{-1}(\pi p/(2\alpha))$. It is routine to see that

$$f(t) - \varepsilon t^p\cos(\pi p/2\alpha) - b \leqslant 0. \tag{2.10}$$

This is clear when $t \leqslant r$, for f is nondecreasing. If $t > r$, say $a^j r \leqslant t \leqslant a^{j+1}r$ $(j = 1, 2, \ldots)$ then (1.4) gives

$$f(t) \leqslant a^{(j+1)p}f(r) \leqslant a^p(t/r)^p f(r) = \varepsilon t^p\cos(\pi p/(2\alpha)),$$

and (2.10) holds. Now (2.10) implies

$$\ln\left|\Phi(te^{\pm i\pi/2\alpha})\right| \leqslant f(t) - \varepsilon t^p\cos(\pi p/2\alpha) - b \leqslant 0,$$

and the Phragmén-Lindelöf principle shows that $\ln|\Phi(z)| \leqslant 0$ $(z \in \Lambda_\alpha)$. This gives (2.9) with $C = a^p\cos^{-1}(\pi p/2\alpha) + 1$.

COROLLARY. *Let F be holomorphic in Λ_α, of order $< \alpha$, and suppose*

$$\ln\left|F(re^{\pm i\pi/2\alpha})\right| = o(\phi(r)) \qquad (r \to \infty)$$

where $\phi \in A(p)$ for some $p < \alpha$. Then

$$\ln|F(z)| = o\big(\phi(|z|)\big) \qquad (z \to \infty,\, z \in \Lambda_\alpha).$$

LEMMA 6. *Let $F(z)$ be holomorphic, of order $\leqslant \rho$ in Λ_α, and have no zeros in Λ_α. Then for any $\varepsilon > 0$, the function $F^{-1}(z)$ has order $\leqslant \rho^* \equiv \max(\rho, \alpha)$ in $\Lambda_{\alpha+\varepsilon}$.*

PROOF. Let $f(\lambda)$ be a holomorphic nonzero in Λ_1 and $|f(\lambda)| < M$. Consider the function $g(\lambda) = \ln M - \ln f(\lambda)$. Since $\operatorname{Re} g(\lambda) > 0$ as $\lambda \in \Lambda_1$, we may apply Carathéodory's inequality (see [**9**, Chapter 1, §7])

$$g(\lambda) < 5|g(1)|r/\sin\theta, \qquad \lambda = re^{i\theta}, |\lambda| > 1.$$

This implies

$$\ln\left|Mf^{-1}(\lambda)\right| \leqslant \left|-\ln Mf^{-1}(\lambda)\right| = \left|\ln Mf^{-1}(\lambda)\right| \leqslant Cr/\sin\theta$$

and hence

$$\ln\left|f^{-1}(\lambda)\right| \leqslant Cr, \qquad \lambda \in \Lambda_{1+\varepsilon}, \varepsilon > 0. \tag{2.11}$$

Suppose $\alpha \geqslant \rho$. Take any $\varepsilon > 0$ and consider the function $G(z) = F(z)\exp(-z^{\alpha+\varepsilon})$, which is bounded in $\Lambda_{\alpha+2\varepsilon}$. Then the function $f(\lambda) = G(\lambda^{1/(\alpha+2\varepsilon)})$ satisfies (2.11) and we have

$$\ln\left|F^{-1}(z)\right| \leqslant C|z|^{\alpha+2\varepsilon}, \qquad z \in \Lambda_{\alpha+3\varepsilon}. \tag{2.12}$$

Now let $\alpha < \rho$. We have from (2.12),

$$\ln\left|F^{-1}(z)\right| \leqslant C|z|^{\rho+\varepsilon}, \qquad z \in \Lambda_{\rho+\varepsilon}. \tag{2.13}$$

But we may obtain (2.13) in any sector $\Lambda^\phi_{\rho+\varepsilon} = \{\lambda\colon |\phi - \arg\lambda| \leqslant \pi/2(\rho+\varepsilon)\}$ if $\Lambda^\phi_{\rho+\varepsilon} \subset \Lambda_{\alpha+\varepsilon}$, hence estimate (2.13) is valid, as $z \in \Lambda_{\alpha+\varepsilon}$. This proves Lemma 2 with (2.12).

3. Estimates for Theorem 1.

LEMMA 7. *The quotient $B^*(z)$ satisfies*

$$\big|\ln|B^*(z)|\big| = o\big(n_g(r)\big) \tag{3.1}$$

as $|z| \to \infty$ in $\Lambda_1 - \Lambda_{1+\delta}$.

PROOF. Let

$$\chi_1(z) = f(z)B_f(z)^{-1}, \quad \chi_2(z) = g(z)B_g(z)^{-1}, \quad \Delta(z) = \chi_1(z)\chi_2(z)^{-1}.$$

We claim that if ρ^* is the order of Δ in Λ_1 then

$$\rho^* < 1. \tag{3.2}$$

Once (3.2) is established, it is clear how to get (3.1). For since $|B_f(iy)| = |B_g(iy)| \equiv 1$ $(-\infty < y < \infty)$, we see from (1.10) that

$$|\ln|\Delta(iy)|| \equiv \left|\ln\left|\frac{f(iy)}{g(iy)}\right|\right| = o(n_g(y)) \qquad (-\infty < y < \infty),$$

so (3.2) allows us to use the corollary to Lemma 5 with $\alpha = 1$, and deduce that

$$|\ln|\Delta(z)|| = o(n_g(r))$$

as $|z| = r \to \infty$ in Δ_1. But now this and (1.10) give the lemma since

$$\pm\ln|B^*(z)| = \mp|\Delta(z)| \pm \ln|f(z)/g(z)|.$$

We now prove (3.2). The functions χ_i are holomorphic and nonzero in $\Lambda_{1-\delta}$. We claim that the order ρ_i of χ_i is $\leqslant 1-\delta$ in $\Lambda_{1-\delta}$; once this is known, (3.2) is a consequence of Lemma 6 with $\rho = 1-\delta$ and $\alpha = 1-\delta$ (so that $\varepsilon = \delta$). Let us consider χ_1 in the region $D(R) = \Lambda_{1-\delta} \cap \{|z| \leqslant R\}$ for certain large R. On $\arg z = \pm\pi/2(1-\delta)$, we recall that f has order $1-\delta$ and $|B_f| \geqslant 1$, and hence

$$|\chi_1(z)\exp(-2z^{(1-\delta+\varepsilon)})| = O(1), \tag{3.3}$$

for any fixed $\varepsilon > 0$. We next obtain a similar estimate on $\Lambda_{1-\delta} \cap \{|z| = R\}$ for an unbounded set of R. Since $T(r, B_f) = O(r^{1-\delta+\varepsilon})$ (see (2.4)) we have from the $\cos\pi\rho$-theorem [**10**, p. 275] that

$$\inf_{|z|=R} \ln|B_f(z)| \geqslant -R^{1-\delta+\varepsilon},$$

for a sequence $R = R_n \to \infty$. Also, f has order $\leqslant 1-\delta$ in $\Lambda_{1-\delta}$, so (3.3) holds on $\Lambda_{1-\delta} \cap \{|z| = R\}$ for these R. The maximum principle now gives (3.3) in all of $\Lambda_{1-\delta}$, so χ_1 has order $\leqslant 1-\delta$ in $\Lambda_{1-\delta}$.

The same holds for χ_2, and according to Lemma 6, χ_2^{-1} has order less than $1-\delta$ in $\Lambda_{1-\delta}$. This establishes (3.2) and completes the proof of Lemma 7.

We now can make a major step toward Theorem 1.

LEMMA 8. *We have* $N_f(r) \leqslant CN_g(r)$, $n_f(r) \leqslant Cn_g(r)$.

PROOF. Recall the definition of $I(\phi, t)$ from this statement of Lemma 4. Then Lemmas 6 and 7 give

$$\int_0^r I\left(\pm\frac{\pi}{2(1+\delta)}, t\right)t^{-1}\,dt = o(n_g(r)). \tag{3.4}$$

We next must estimate $\int_\psi^\theta \ln|B^*(re^{i\phi})|\,d\phi$ from above. Since $|B_f| \leqslant 1$ in Λ_1, it suffices to obtain upper bounds for $-\int_\psi^\theta \ln|B_g(re^{i\phi})|\,d\phi$; this is fortunate since Lemma 2 gives good information on $T(r, B_g)$. A useful lemma of Edrei and Fuchs [**5**, p. 322], (2.3) and (1.6) yield that if I is a θ-interval of length $|I|$, then

$$\int_I \left|\ln|B_g(re^{i\phi})|\right| d\phi \leqslant \frac{25R}{R-r}T(R, B_g)|I|\left\{1 + \log^+\frac{1}{|I|}\right\} \tag{3.5}$$

$$\leqslant Cn_g(r)|I|\left\{1 + \log^+|I|^{-1}\right\} \leqslant Cn_g(r)$$

if $R = 2r > r_0$. Estimates (3.4) and (3.5) may be inserted to (2.7), leading to

$$N_f(r) \leqslant N_g(r) + o(n_g(r)) + Cn_g(r)$$

and an appeal to (1.12) and (1.13) gives

$$N_f(r) \leqslant CN_g(r), \qquad n_f(r) \leqslant Cn_g(r). \tag{3.6}$$

It remains to reverse the inequalities of (3.6). We first improve the estimate of $T(r, B_f)$ from (2.4).

LEMMA 9. *The characteristic of B_f satisfies*

$$T(r, B_f) = O(n_g(r)). \tag{3.7}$$

Further if $r_0 < r < R/2$, we have

$$T(r, B_f) \leqslant CN_f(R) + C(r/R)^{1-p}N_g(r), \tag{3.8}$$

where $p < 1$ is the class of $A(p)$ to which $n_g(r)$ belongs.

PROOF. Write $B_f = P_1(f)P_2(f)^{-1}$ as in the proof of Lemma 2, where now the z_k are the zeros of f. The information in (3.6) allows the estimates made for $T(r, P_i)$ $(i = 1, 2)$ in Lemma 2 to be transferred to $T(r, P_1(z))$ and $T(r, P_2(z))$. Thus $T(r, P_i(z)) = O(n_g(r))$, and (3.7) follows.

An approximation lemma of Edrei and Fuchs [**4**, p. 296, with $q = 0$] shows that

$$\ln|P_1(z)| = \log \prod_{|z_k| \leqslant R} \left|1 - \frac{z}{z_k}\right| + S(z, R) \qquad \left(|z| = r < \tfrac{1}{2}R\right) \tag{3.9}$$

where the remainder S satisfies

$$|S(z, R)| \leqslant 14(r/R)T(2R, P_1). \tag{3.10}$$

Let us once more compute

$$T(r, P_1) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+|P_1(re^{i\theta})|\,d\theta.$$

Once again, as in [**7**, p. 27] (cf. our proof of Lemma 2) we have

$$\int_0^{2\pi} \sum_{|z_k| \leqslant R} \ln^+|1 - z_k^{-1}re^{i\theta}|\,d\theta = O(N_f(R)),$$

and since (3.7) implies that $T(r, P_1) = O(N_g(r))$, (1.6) and (3.10) show that

$$|S(z, R)| \leqslant C(r/R)^{1-p}N_g(r) \quad \text{if } R > 2r.$$

This proves (3.8).

LEMMA 10. *We have $N_g(r) \leqslant CN_f(r)$.*

PROOF. Since $\ln|B_g(z)| \leqslant 0$ for $z \in \Lambda_1$, (3.4) and (2.7) show

$$N_g(r) \leqslant N_f(r) + \varepsilon n_g(r) + C \max_{-\pi/2 \leqslant \psi < \theta \leqslant \pi/2} \left[-\int_\psi^\theta \ln|B_f(fe^{i\phi})|\,d\phi \right]. \tag{3.11}$$

According to the Edrei-Fuchs lemma [**3**, p. 322] which was already used in (3.5), and our refined estimate (3.8) of $T(r, B_f)$, we see that

$$\int_\psi^\theta \left|\ln\left|B_f(re^{i\phi})\right|\right| d\phi \leqslant C\frac{R}{R-r}\left\{N_f(R') + \left(\frac{R}{R'}\right)^{1-p} N_g(R)\right\}$$

uniformly in $-\pi/2 \leqslant \psi < \theta \leqslant \pi/2$, where $r \leqslant R/2 \leqslant R'/4$. Given $\varepsilon > 0$, take $R = 2r$, and $R' = cr$ so large that (uniformly in ψ and θ)

$$\int_\psi^\theta \left|\ln\left|B_f(re^{i\phi})\right|\right| d\phi \leqslant CN_f(R') + \varepsilon N_g(r);$$

this is possible since $p < 1$, $N_g(r)$ satisfies (1.6) and we know (3.7). Now (3.11) gives $(1-\varepsilon)N_g(r) \leqslant CN_f(R')$. But $N_g(R') = N_g(cr) \leqslant CN_g(r)$, hence $N_g(R') \leqslant CN(R')$ and Lemma 10 follows.

A combination of Lemmas 8, 10 and Note 2 gives $N_f(r) \asymp N_g(r) \asymp n_g(r)$. This implies that $N_f(r) \in A \cap B$, hence $n_f(r) \asymp N_f(r)$ and $n_f(r) \asymp n_g(r)$. Theorem 1 is proved.

4. Theorem on $\sim$ equivalence. The assumption of Theorem 1 does not allow us to deduce $N_f \sim N_g$ or $n_f \sim n_g$. As an example we consider the functions $f(z) = \sin z + e^{\varepsilon z}$, $g(z) = \sin z$. We have $n_g(r) \in B \cap A_p$, $p = 1$, and condition (1.10) holds uniformly, as $z \to \infty$ in $\Lambda_1 - \Lambda_\gamma$, $1 < \gamma < (\pi/2)\operatorname{arctg}\varepsilon$, but $n_g(r) = r/\pi + O(1)$, $n_1(r) = \sqrt{1+\varepsilon^2}\, r/\pi + O(1)$. The example of these functions shows that we can expect $n_f \sim n_g$ when the zeros of f and g asymptotically are contained in an arbitrary small sector and condition (1.10) holds outside of that sector.

THEOREM 2. *Let f and g be holomorphic of finite order $\leqslant \beta$ in the sector Λ_{α_0}, and for any $\alpha > \alpha_0$ all their zeros with the exception of a finite number are contained in Λ_α. Suppose that*

$$\left|\ln|f(z)/g(z)|\right| = O\big(n_g(r)\big) \tag{4.1}$$

uniformly as $z \to \infty$ in $\Lambda_{\alpha_0} - \Lambda_\alpha$.

Then $N_g(r) \sim N_f(r)$, if $n_g(r) \in A \cap B$, and $n_g(r) \sim n_f(r)$ if $n_g(r) \in C \cap B$.

PROOF. Since $n_g(r) \in A$ we can find p such that $n_g(r) \in A_p$. If $\gamma = \max(\alpha_0, \beta p)$ and $\alpha > \gamma$, then (4.1) holds in $\Lambda_\gamma - \Lambda_\alpha$, and according to Theorem 1 we have

$$n_f(r) \asymp n_g(r). \tag{4.2}$$

We may assume $\gamma < 1$, otherwise we ought to replace z by $z^{(\gamma+\tau)^{-1}}$, $\tau > 0$.

As in Theorem 1 we have to estimate the terms in (2.7). According to (4.1) the first term on the right side of (2.7) has the estimate $o(N_g(r))$. Using (3.5), (4.1) and (4.2), we see also that

$$\int_\theta^\psi \ln|B^*(re^{i\phi})|\,d\phi = \left(\int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\theta} + \int_{\psi}^{-\varepsilon}\right)\ln|B^*(re^{i\phi})|\,d\phi$$

$$\leqslant C\varepsilon\left(1 + \log^+\frac{1}{\varepsilon}\right)N_g(r) + \left(\int_\varepsilon^\psi + \int_\theta^{-\varepsilon}\right)\ln|B^*(re^{i\phi})|\,d\phi = o\big(N_g(r)\big).$$

Consequently, the second term on the right side of (2.7) has the estimate $o(N_g(r))$, and according to definition (1.3) we have

$$N_g(r) \sim N_f(r). \tag{4.3}$$

Now we deduce from (4.3) that $n_f \sim n_g$ if $n_g(r) \in B \cap C$.

Given $\varepsilon > 0$ take $\delta > 0$ such that

$$n_g((1+\delta)r) \leqslant (1+\varepsilon)n_g(r). \tag{4.4}$$

We have

$$\begin{aligned} \big[n_f(r) - n_g((1+\delta)r)\big]\ln(1+\delta) &\leqslant \int_r^{(1+\delta)r} \frac{n_f(t)-n_g(t)}{t}\,dt \\ &= N_f((1+\delta)r) - N_g((1+\delta)r) + N_f(r) - N_g(r) \\ &= o\big(N_g((1+\delta)r)\big) = o\big(n_g(r)\big). \end{aligned} \tag{4.5}$$

Hence, from (4.4) we obtain

$$n_f(r) \leqslant n_g((1+\delta)r)[1+o(1)] \leqslant (1+\varepsilon)n_g(r). \tag{4.6}$$

As in (4.5) we have

$$\big[n_g(r) - n_f(r(1+\delta))\big]\ln(1+\delta) \leqslant \int_r^{(1+\delta)r} \frac{n_g(t)-n_f(t)}{t}\,dt = o\big(n_g(t)\big).$$

Consequently,

$$n_f(r(1+\delta)) \geqslant (1-\varepsilon)n_g(r(1+\delta)). \tag{4.7}$$

The combination of (4.6) and (4.7) gives Theorem 2.

5. Asymptotic behaviour of eigenvalues of certain perturbed normal operators. The first general result on the distribution of eigenvalues of an operator $G = H(I+S)$, where H, S are compact operators and $H > 0$, was established by M. V. Keldysh **[8]** (see also [**6**, Chapter V, §11]). He proved

$$\sum_{\lambda_k^{-1} < r} 1 = n_H(r) \sim n_G(r) = \sum_{|\mu_k^{-1}| < r} 1,$$

where λ_k, μ_k are the eigenvalues of operators H and G, respectively, provided that the function $n_H(r)$ satisfies some tauberian conditions.

This result has been generalized by many authors. The references can be found in **[2]** (see also **[1]**). As an application of Theorem 2 we will prove a result concerning this problem, which covers those of many authors.[1]

In this section we denote by σ_∞ the collection of all compact operators and by R the collection of all bounded operators acting in Hilbert space $\mathfrak{H}$. Also by $\lambda_k(A)$ we

[1]After submitting this paper for publication the author discovered a more complete and general result announced by A. S. Markus and V. I. Macaev in Functional Anal. Appl. (**13** (1979), pp. 93–94 (Russian)). Using another method G. V. Radzievskii also obtained a result similar to Theorem 3 (Mat. Sb. **112** (1980), pp. 396–420 (Russian)).

denote the eigenvalues of an operator A and by $s_k(A)$ the s-numbers of this operator, i.e. the eigenvalues of $(AA^*)^{1/2}$. If $\Sigma s_k(A) < \infty$ $(A \in \sigma_1)$, the characteristic determinant of the operator A (see [6, Chapter 4, §1]),

$$\mathfrak{D}_A(\mu) = \det(I - \mu A) = \prod_{k=1}^{\infty} (1 - \mu\lambda_k(A)),$$

converges. If $A(\mu)$ is a holomorphic operator-valued function in the domain $\Lambda \in \mathbf{C}$ with values in σ_1, then the function

$$\det(I - A(\mu)) = \prod_{k=1}^{\infty} (1 - \lambda_k(A(\mu)))$$

is holomorphic in the same domain [6, Chapter IV, §1]. Furthermore, if $(I - A(\mu_0))^{-1}$ for at least one point $\mu_0 \in \Lambda$, then from the results of [3, Chapter 1, §5] we conclude that $(I - A(\mu))^{-1} \in R$ for all $\mu \in \Lambda$ with the possible exception of certain isolated points $\{\mu_k\}$, and $\dim \mathfrak{L}_k = m_k < \infty$, where $\mathfrak{L}_k$ is the subspace consisting of eigen and associated vectors corresponding to the eigenvalue μ_k. These points are m_k-multiple zeros of the function $\det(I - A(\mu))$.

Now let H be a normal compact operator in $\mathfrak{H}$. Suppose that in a certain sector Λ in the complex plane the characteristic numbers of operator H, $\mu_k(H) = \lambda_k^{-1}(H)$, are concentrated asymptotically along a ray $\gamma \in \Lambda$. We assume that γ is the positive semiaxis; then our hypothesis means that not more than a finite number of $\{\mu_k(H)\}$ are contained in the domain $\Lambda_{\alpha_0} \setminus \Lambda_\alpha$ for some α_0 and any $\alpha > \alpha_0$.

With operator H we connect the operator-valued function

$$L(\mu) = I - T - A(\mu) - \mu H, \tag{5.1}$$

where I is the identity operator, $T \in \sigma_\infty$, $A(\mu)$ is a holomorphic operator-valued function in the sector Λ_{α_0} with values in R and $\|A(\mu)\| \to 0$ as $\mu \to \infty$ in Λ_{α_0}. Suppose also $(I - T)^{-1} \in R$.

Take any $\alpha > \alpha_0$ and consider the functions

$$n_H(\alpha, r) = \sum_{\substack{\mu_k(H) \leqslant r \\ \mu_k(H) \in \Lambda_\alpha}} 1; \qquad n_L(\alpha, r) = \sum_{\substack{\mu_k(L) \leqslant r \\ \mu_k(L) \in \Lambda_\alpha}} 1. \tag{5.2}$$

We suppose that a number $\mu_k(H)$ or $\mu_k(L)$ is repeated in (5.2) s times, if it is an s-multiple eigenvalue of H or L, respectively.

THEOREM 3. *Let H be a compact normal operator and $n_H(\alpha_0, r) - n_H(\alpha, r) \leqslant C(\alpha)$ for some α_0 and any $\alpha > \alpha_0$. If $L(\mu)$ is defined by* (5.1) *and $n_H(\alpha_0, r) \in C \cap B$, then*

$$n_H(\alpha, r) \sim n_L(\alpha, r), \qquad \alpha > \alpha_0.$$

PROOF. Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ where $\mathfrak{H}_1$ is the subspace consisting of all eigenvalues of the operator H corresponding to $\{\mu_k(H)\} \in \Lambda_{\alpha_0}$. Define by P the projector $P\mathfrak{H} = \mathfrak{H}_1$ and $Q = I - P$. Since H is a normal operator, we have $PQ = QP = 0$.

Now we represent the operator-valued function $L(\mu)$ in the form

$$\begin{aligned} L(\mu) &= I - T - A(\mu) - \mu(P+Q)H \\ &= \left[I - T(Q+P)(I-\mu QH)^{-1} - A(\mu)(I-\mu QH)^{-1} \right. \\ &\qquad \left. - \mu PH(I-\mu QH)^{-1}\right](I-\mu QH) \\ &= \left[I - TP - TQ(I-\mu QH)^{-1} - A(\mu)(I-\mu QH)^{-1} - \mu PH\right](I-\mu QH). \end{aligned}$$

Since $(I-T)^{-1} \in R$, we have $(I-TP)^{-1} \in R$. The operator QH is a normal operator and $\{\mu_k(QH)\} \in \Lambda_{\alpha_0}$, hence (see [**6**, Chapter 5, §7])

$$\|(I-\mu QH)^{-1}\| \leqslant \sin^{-1} \tfrac{\pi}{2}(1/\alpha_0 - 1/\alpha), \tag{5.3}$$

if $\mu \in \Lambda_\alpha$, $\alpha > \alpha_0$.

Let $T \in \sigma_\infty$; then

$$\|TQ(I-\mu QH)^{-1}\| \to 0 \tag{5.4}$$

uniformly as $\mu \to \infty$ in Λ_α. This assertion is proved in [**6**, Chapter 5, §7], if $Q = I$ and $H > 0$. But absolutely the same method allows us to obtain (4.4). Using (4.3) and (4.4) we represent $L(\mu)$ in the form

$$L(\mu) = (I - T_1 - A_1(\mu))^{-1}(I - \mu(I - T_1 - A_1(\mu))PH)(I-\mu QH) \tag{5.5}$$

where

$$I - T_1 - A_1(\mu) = (I - TP - A_2(\mu))^{-1},$$
$$A_2(\mu) = TQ(I-\mu QH)^{-1} - A(\mu)(I-\mu QH)^{-1},$$

$T_1 \in \sigma_\infty$ and $\|A_i(\mu)\| \to 0$, $i = 1, 2$, as $\mu \to \infty$ in Λ_α.

Define $H_1 = PH$ and

$$L_1(\mu) = I - \mu(I - T_1 - A_1(\mu))H_1. \tag{5.6}$$

First we consider the case $n_H(\alpha_0, r) = n_{H_1}(r) \in A_p$, $p < 1$. In this case $H_1 \in \sigma_1$ and the functions $\det(I - \mu H_1)$ and $\det L_1(\mu)$ are well defined and holomorphic in Λ_α. If $A, B \in \sigma_1$, then (see [**6**, Chapter 4, §3])

$$\det(I-A)(I-B)^{-1} = \det(I-A)/\det(I-B).$$

Using this formula and (5.6) we obtain

$$\det L_1(\mu)/\det(I-\mu H_1) = \det(I - \mu K(\mu)), \tag{5.7}$$

$$\begin{aligned} \det(I-\mu H_1)/\det L_1(\mu) &= \det(I-\mu K(\mu))^{-1} \\ &= \det\left(I + \mu K(\mu)(I-\mu K(\mu))^{-1}\right) \end{aligned} \tag{5.8}$$

where

$$K(\mu) = (T_1 + A_1(\mu))(I-\mu H_1)^{-1}H_1. \tag{5.9}$$

Now we recall some properties of s-numbers of compact operators (see [**6**, Chapter 2, §2.3]).

We have

$$s_j(BA) \leqslant \|B\| s_j(A), \qquad s_j(AB) \leqslant \|B\| s_j(A), \tag{5.10}$$

if $A \in \sigma_\infty$ and $B \in R$,

$$s_{j+\kappa}(BA) \leqslant s_{\kappa+1}(B)s_j(A), \tag{5.11}$$

if $A \in \sigma_\infty$, $B \in \sigma_\infty$ and $j \geqslant +1$.

For $A \in \sigma_1$ we also have

$$\left|\prod_{j=1}^{\infty}\big(1+\lambda_j(A)\big)\right| \leqslant \prod_{j=1}^{\infty}\big(1+s_j(A)\big). \tag{5.12}$$

Since H_1 is a normal operator and for any $\beta < \infty$ the set $\{\mu_k(H_1)\} \in \Lambda_\beta$ with the exception of a finite number of $\mu_k(H_1)$, we obtain

$$\left\|(I-\mu H_1)^{-1}\right\| \leqslant \sin^{-1}\frac{\pi}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right) \leqslant \sin^{-1}\frac{\pi}{4\alpha} \tag{5.13}$$

if $|\mu| > r_0$ and $\beta/2 > \alpha > \alpha_0$.

Given $\varepsilon > 0$, take integer κ such that $s_{\kappa+1}(T_1) \leqslant \varepsilon \sin \pi/4\alpha$. Then, using (5.10), (5.11) and recalling that $\|A_1(\mu)\| \to 0$ as $\mu \to \infty$ in Λ_{α_0}, we obtain from (5.9):

$$s_{j+\kappa}(K(\mu)) \leqslant 2\varepsilon s_j(H_1), \qquad |\mu| > r_0,\ \mu \in \Lambda_{\alpha_0} - \Lambda_\beta. \tag{5.14}$$

Now (5.12) implies

$$|\det(I-\mu K(\mu))| \leqslant C|\mu|^\kappa \prod_{j=1}^{\infty}\big(1+2\varepsilon|\mu|s_j(H_1)\big). \tag{5.15}$$

We claim

$$\left\|(I-\mu K(\mu))^{-1}\right\| \leqslant C \tag{5.16}$$

as $\mu \in \Lambda_\alpha - \Lambda_\beta$, $\beta > \alpha > \alpha_0$ and $|\mu| > r_0$. Once this is known, we obtain from (5.9) and (5.14):

$$\left|\det(I-\mu K(\mu))^{-1}\right| \leqslant C|\mu|^\kappa \prod_{j=1}^{\infty}\big(1+\varepsilon|\mu|s_j(H_1)\big). \tag{5.17}$$

Once again as in Lemma 9 (see (3.9) and (3.10)), we have, using our hypothesis $n_{H_1}(r) \in A_p$, $p < 1$, that

$$\left|\sum_{j=1}^{\infty}\ln\left(1-\frac{\mu}{\mu_j(H_1)}\right)\right| \leqslant \sum_{j=1}^{\infty}\ln\left(1+\frac{|\mu|}{|\mu_j(H_1)|}\right) = O\big(n_{H_1}(r)\big).$$

Then (5.15) and (5.17) give

$$\begin{aligned} \pm\ln|\det L_1(\mu)/\det(I-\mu H_1)| &= \pm\ln|\det(I-\mu K(\mu))| \\ &\leqslant C + r^\kappa + \sum_{j=1}^{\infty}\ln\big(1+\varepsilon r s_j(H_1)\big) \\ &= C + r^\kappa + \sum_{j=1}^{\infty}\ln\left(1+\frac{\varepsilon r}{|\mu_j(H_1)|}\right) \\ &= O\big(n_{H_1}(\varepsilon r)\big) = o\big(n_{H_1}(r)\big) \end{aligned} \tag{5.18}$$

as $|\mu| = r \to \infty$ and $\mu \in \Lambda_\alpha \setminus \Lambda_\beta$, $\beta > \alpha > \alpha_0$.

The last estimate shows that we can apply Theorem 2 and deduce $n_{L_1}(\alpha, r) \sim n_{H_1}(r)$. But (5.5) and (5.6) imply that $|n_L(\alpha, r) - n_{L_1}(\alpha, r)| \leqslant C$, hence $n_{L_1}(\alpha, r) \sim n_H(\alpha, r)$ and Theorem 3 follows.

Now prove (5.16). We have from (5.5), (5.6) and (5.9):

(5.19)

$$\begin{aligned} I - \mu K(\mu) &= L_1(\mu)(I - \mu H_1)^{-1} \\ &= (I - T_1 - A_1(\mu))(I - TP - A_2(\mu) - \mu H_1)(I - \mu H_1)^{-1} \\ &= (I - T_1 - A_1(\mu))\big(I - TP(I - \mu PH)^{-1} - A_2(\mu)(I - \mu H_1)^{-1}\big). \end{aligned}$$

In (5.4) replace Q by P and recall (5.13). Then (5.16) follows from (5.19).

We proved Theorem 3 under hypothesis $n_{H_1}(r) \in A_p$, $p < 1$. If $p > 1$, then take integer $l > p$ and consider instead of $L_1(\mu)$ the operator-valued function

$$\begin{aligned} L_l(\mu) &= I - \mu^l(I - T_1 - A_1(\mu)H_1)^l \\ &= \prod_{k=1}^{l} (I - w_k\mu(I - T_1 - A_1(\mu)H_1), \end{aligned} \tag{5.20}$$

where $w_k^l = 1$, $k = 1, \ldots, l$. It follows from (5.20) that $|n_{L_1}(\alpha, r) - n_{L_l}(\alpha, r)| < C$ if $\alpha < \alpha_1 = \max(\alpha_0, l)$. The functions $\det L_l(\mu)$ and $\det(I - \mu^l H_1^l)$ are well defined and holomorphic in Λ_{α_1}. As before, we have to show

$$\pm \ln\big|\det L_l(\mu)/\det\big(I - \mu^l H_1^l\big)\big| = o\big(n_{H_1}(r)\big), \tag{5.21}$$

as $\mu \to \infty$ in $\Lambda_{\alpha_1} \setminus \Lambda_\alpha$ for any $\alpha > \alpha_1$, and then to apply Theorem 2. But it is clear that we may obtain (5.21) in the same way as (5.18). This completes the proof of Theorem 3.

COROLLARY. *Let H be a compact normal operator, whose characteristic numbers $\mu_k(H)$ lie on some rays $\gamma_1, \ldots, \gamma_m$ in the complex plane and $n_H(\gamma_j, r) \in B \cap C$, $j = 1, \ldots, m$.*

Let $L(\mu)$ be defined by (5.1) *and $A(\mu)$ be a holomorphic function in the neighborhood of ∞, such that $A(\infty) = 0$. Then the eigenvalues of $L(\mu)$ are concentrated asymptotically along the rays $\gamma_1, \ldots, \gamma_m$ and*

$$n_H(\gamma_j, r) \sim n_L(\gamma_j, r), \qquad j = 1, \ldots, m$$

(*we define $n_L(\gamma_j, r)$ by* (5.2), *where Λ_α is replaced by some small sector Λ containing γ_j*).

Note 1. Theorem 3 will be valid if we assume in (5.1) (see (5.6))

$$\|A(\mu)H^{-1}\| = o(|\mu|), \qquad \mu \to \infty, \mu \in \Lambda_{\alpha_0},$$

instead of $\|A(\mu)\| = o(1)$.

Note 2. If $H > 0$, $\|A(\mu)H^{-1}\| = O(1)$ as $\mu \to \infty$ and $n_H(r)$ satisfies some tauberian conditions (those tauberian conditions involve $n_H(r) \in B \cap C$) then the assertion of Theorem 3 was announced in **[1]**.

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